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TITLE OF THESIS... Quotient Resolvents of Various Classes of Topological Spaces

UNIVERSITY... Alberta

DEGREE FOR WHICH THESIS WAS PRESENTED.. Ph.D.

YEAR THIS DEGREE GRANTED.... 1973

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QUOTIENT RESOLVANTS OF VARIOUS CLASSES OF TOPOLOGICAL SPACES

by

© BRIAN DOUGLAS WARRACK

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1973

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled QUOTIENT RESOLVANTS OF VARIOUS CLASSES OF TOPOLOGICAL SPACES submitted by BRIAN DOUGLAS WARRACK in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

Given a class $\mathcal{B}$ of topological spaces and a class F of mappings of topological spaces, the F-resolvant of $\mathcal{B}$ is defined to be the class $R_F(\mathcal{B})$ of topological spaces whose every F-image lies in $\mathcal{B}$. Whenever F is closed under composition and includes identity maps, $R_F(\mathcal{B})$ is easily seen to be the largest class of spaces smaller than $\mathcal{B}$ which is closed under F-maps.

The class $R_F(\mathcal{B})$ was recently isolated as an object of study by MacDonald and Willard. Our main interest in this thesis is to investigate the F-resolvant of various classes of topological spaces for each of five classes F of quotient mappings: the closed, the open, the bi-quotient, the hereditarily quotient and the quotient mappings. Also considered are the cases in which F is either the class of continuous mappings or the class of perfect mappings.

After introducing the resolvant problem in Chapter I, the F-resolvants of first and second countable spaces and of metrizable spaces are considered in Chapter II. Chapter III is devoted to F-resolvants of the class $\mathcal{L}$ of locally compact spaces. It is shown that whenever F represents either the class of closed maps or hereditarily quotient maps, then $R_F(\mathcal{L})$ is the collection of all normal spaces which can be written $X = X_0 \oplus D$, where $|\beta X_0 - X_0| < \aleph_0$ and D consists of points isolated in X.

Resolvants of paracompact spaces are considered in Chapter IV, where it is established that every Hausdorff quotient of a paracompact

space X of point-countable type is paracompact iff the set $\operatorname{acc} X$ of accumulation points of X is compact, or else X is locally compact and $\operatorname{acc} X$ is Lindelöf. Similar results are obtained for the open and bi-quotient resolvants in the case of 0-dimensional paracompact spaces. Also given is an example of a space X whose every Hausdorff quotient is paracompact although $\operatorname{acc} X$ is non-Lindelöf, resolving in the negative a conjecture of MacDonald and Willard.

Collected together in the last chapter are results, largely proved in previous chapters, concerning the F-resolvants of regular and normal spaces.

ACKNOWLEDGEMENTS

The author gratefully acknowledges partial support received from the National Research Council of Canada and from the University of Alberta.

I would like to express my sincere gratitude to my supervisor, Dr. Stephen Willard, for suggesting the topic of this thesis and providing guidance during its development, and for offering encouragement whenever such was needed.

Particular thanks is due my friend and former teacher, Dr. S.A. Naimpally, who first introduced me to general topology and who has maintained an interest in my studies ever since. Thanks also to Ms. Olwyn Buckland who typed this thesis both quickly and accurately.

I wish to pay special tribute to my parents, to whom this thesis is dedicated, for having provided such a generous amount of encouragement and support throughout my years of schooling. Your continued faith in your son has been deeply appreciated.

Above all, a very special thanks to my wife Leslie for her patience, understanding, and the many sacrifices she has made in order that I could devote more time to my studies.

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CHAPTER I

INTRODUCTION

1. Background.

Much of the recent research in general topology has been stimulated by Arhangel'skii's 1966 survey article entitled "Mappings and Spaces" [6], in which it was suggested that greater use be made of the method of mutual classifications of mappings and spaces in providing a unified approach to the study of topology. Among the problems which he isolated for investigation are the following, which were first formulated by Alexandroff [1] in his report to the 1961 Prague Symposium on general topology.

Given a class $\mathcal{B}$ of topological spaces and a class F of mappings of topological spaces,

(1) characterize the class $P_F(\mathcal{B})$ of spaces which are F-images of of spaces from $\mathcal{B}$;

(2) characterize the class of spaces which are F-preimages of spaces from $\mathcal{B}$.

Overlooked in Arhangel'skii's article, but singled out for attention in [22] and [23], was the problem with which we are concerned in this thesis:

(3) characterize the class $R_F(\mathcal{B})$ of spaces whose every F-image

lies in $\mathcal{B}$.

For all classes F of mappings considered here, F is closed under composition and includes identity maps. It is easily checked that whenever this is the case, then $P_F(\mathcal{B})$, called the F-projector of $\mathcal{B}$, is the smallest class of spaces larger than $\mathcal{B}$ which is closed under F-maps. Similarly $R_F(\mathcal{B})$, called the F-resolvant of $\mathcal{B}$, is the largest class of spaces smaller than $\mathcal{B}$ which is closed under F-maps.[1] Thus we can write $R_F(\mathcal{B}) \subset \mathcal{B} \subset P_F(\mathcal{B})$. Observe that equality on either side implies that $\mathcal{B}$ is closed under F-maps, whence we obtain $R_F(\mathcal{B}) = \mathcal{B} = P_F(\mathcal{B})$.

Our main interest in this thesis is to investigate the F-resolvant of various classes of topological spaces for each of five classes F of quotient mappings: the closed, the open, the bi-quotient, the hereditarily quotient and the quotient mappings. Also considered are the (in a sense, limiting) cases in which F is either the class of continuous mappings or the class of perfect mappings.

The extent to which success has been achieved in obtaining a characterization of various resolvants is summarized in the accompanying table. Each entry (with the exception of those containing a parenthetical rider) characterizes $R_F(\mathcal{B})$ for the class F of mappings of the type given at the beginning of the row and the class $\mathcal{B}$ of spaces of the type given at the top of the column. Whenever a characterization of $R_F(\mathcal{B})$ is not available for $\mathcal{B}$ in general, we have entered a characterization of those spaces from some subclass $\mathcal{B}'$ of $\mathcal{B}$ which belong to $R_F(\mathcal{B})$. The property defining this subclass $\mathcal{B}'$ is noted in parentheses. Thus, for example, entry (3,1) can be read: every open (Hausdorff) image of a

0-dimensional space X of point-countable type is paracompact iff acc X is compact, or else X is locally compact and acc X is Lindelöf. For use in the table only, we define a space X to be an AA-space iff X is a normal space which can be written $X = X_o \oplus D$, where $|\beta X_o - X_o| < \mathcal{N}_o$ and D consists of points isolated in X.

The problem of characterizing projectors has received a good deal of attention since Arhangel'skii's paper appeared, and Michael [29] has recently consolidated investigations in this area. Since we are ultimately interested in determining the relationships between $\mathcal{B}$, $P_F(\mathcal{B})$ and $R_F(\mathcal{B})$, known results concerning the F-projectors of the class of topological spaces in question will be briefly outlined at the beginning of each chapter. Observe that for every class $\mathcal{B}$ of spaces considered here (except second countable spaces), the continuous projector of $\mathcal{B}$ is the collection of all topological spaces, since every space is a continuous image of a discrete space.

Two additional related problems are the following, where F and G represent classes of mappings:

(4) characterize the class of spaces X for which every mapping in F with domain X belongs to G;

(5) characterize the class of spaces Y for which every mapping in F with range Y belongs to G.

Results concerning Problem (4) have been obtained by Dickman and Zame [12], Siwiec [37], Strong [40] and Willard [45]. Problem (5) has been considered, for various classes of mappings and spaces, by Olson [33], Palais

map \ range of map	paracompact	metrizable	locally compact	first countable	second countable
perfect	paracompact	metrizable	locally compact	every compact subset of countable character	second countable
closed	paracompact	metrizable acc X compact	AA - space	every closed subset of countable character	second countable acc X compact
open	acc X compact, or X locally compact & acc X Lindelöf (0-dimensional point-countable type)	metrizable; acc X compact, or X locally compact & acc X separable (0-dimensional)	locally compact	first countable	second countable
bi-quotient	acc X compact, or X locally compact & acc X Lindelöf (0-dimensional point-countable type)	metrizable; acc X compact, or X locally compact & acc X separable (0-dimensional)	locally compact		second countable
hereditarily quotient		metrizable acc X compact	AA - space	every closed subset of countable character	second countable acc X compact
quotient	acc X compact, or X locally compact & acc X Lindelöf (point-countable type)	metrizable acc X compact	AA - space (sequential)	every closed subset of countable character	second countable acc X compact
continuous	compact	compact metrizable	compact	compact; every closed subset of countable character	compact second countable

[34], Siwiec [37], Siwiec and Mancuso [38] and Whyburn [41]. Problems (4) and (5) per se fall outside the scope of this thesis, and will be referred to only when they have a direct bearing on the resolvant problem.

2. Terminology and Notation.

All spaces considered in this thesis are assumed to be Hausdorff topological spaces. When speaking of a topological space (X,τ), we shall omit reference to the underlying topology and simply write X. The term mapping will be used throughout to mean a continuous surjection.

Our notation is generally standard. If A is a subset of a space X, then the restriction of a mapping f to A will be written $f|_A$, the cardinality of A will be denoted by $|A|$, and Int A will mean the interior of A. Following MacDonald and Willard [23], acc X will be used to denote the set of non-isolated (or accumulation) points of a space X. (Some authors refer to acc X as the derived set of X.) N will always represent the set of positive integers, and by $X \oplus Y$ is meant the disjoint union of the spaces X and Y. The abbreviated form (x_n) will often be used to refer to the sequence $(x_n \mid n \in N)$, and "convergent sequence" will be understood to mean the range of the sequence together with its limit point. The first uncountable ordinal will be denoted by ω_1, and by the space of countable ordinals will be meant the interval $[0,\omega_1)$ with the usual order topology. Use of the continuum hypothesis in proving a particular result will be indicated by [CH].

By "the quotient resolvant of paracompact spaces", we shall mean $\mathcal{R}_F(\mathcal{S})$ where $\mathcal{S}$ is the class of all paracompact spaces and F is the

class of all quotient mappings with Hausdorff range. Similar terminology will be used in the case of projectors.

Definitions of topological terms used can, for the most part, be found in Willard [44]. Defined below are some exceptions to this rule.

Definitions 2.1.

(a) A family $\{A_s \mid s \in S\}$ of subsets of a space X is called closure-preserving [26] if $\bigcup_{s\in T} \overline{A}_s = \overline{\bigcup_{s\in T} A_s}$ for every $T \subset S$.

(b) A space X is said to be collectionwise normal [9] if for every discrete family $\{F_s \mid s \in S\}$ of closed subsets of X, there exists a discrete family $\{U_s \mid s \in S\}$ of open sets such that $F_s \subset U_s$ for every $s \in S$.

(c) A space X is locally countably compact if every point in X has a countably compact neighbourhood.

(d) A subspace A of a space X is clopen if it is both open and closed in X.

Definitions 2.2. Let f be a mapping from X to Y.

(a) f is closed if the image of each closed subset of X is a closed subset of Y.

(b) f is perfect if f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

(c) f is open if the image of each open subset of X is an open subset of Y.

(d) f is P_2 (McDougle [24,25]) if for each $y \in Y$, there exists a compact subset K_y of $f^{-1}(y)$ such that $y \in \mathrm{Int}(f(U))$ whenever U is a neighbourhood of K_y.

(e) f is bi-quotient (Michael [27]; see also Hájek [17]) if for every $y \in Y$ and every open cover $\mathcal{U}$ of $f^{-1}(y)$, there exist finitely many members of $\mathcal{U}$ such that y is interior to the image of their union.

(f) f is countably bi-quotient (Siwiec and Mancuso [38]) if for every $y \in Y$ and every countable open cover $\mathcal{U}$ of $f^{-1}(y)$, there exist finitely many members of $\mathcal{U}$ such that y is interior to the image of their union.

(g) f is hereditarily quotient if $f\big|_{f^{-1}(S)}$ is a quotient mapping for every $S \subset Y$.

(h) f is pseudo-open (Arhangel'skii [4]; see also McDougle [24]) if for every $y \in Y$ and every open neighbourhood U of $f^{-1}(y)$, $y \in \mathrm{Int}\, f(U)$.

(i) f is quotient if a subset $S \subset Y$ is open in Y whenever $f^{-1}(S)$ is open in X.

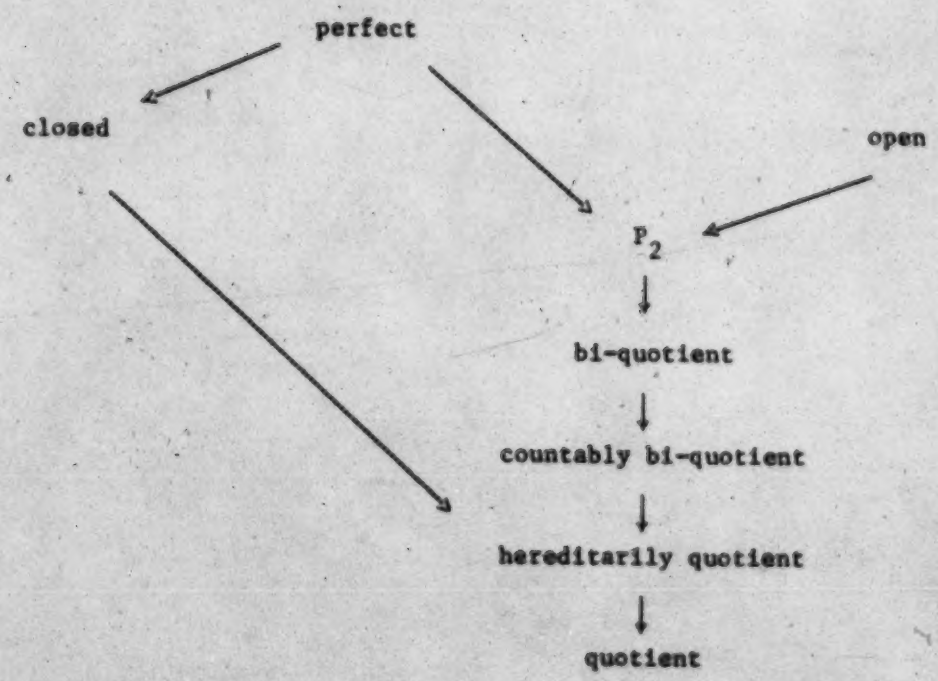
Remarks 2.3. (a) If F represents the class of all mappings of one of the types defined in 2.2 (with the exception of P_2 - maps), then F is closed under composition and includes identity maps. Hence $R_F(\mathcal{B}) \subset \mathcal{B} \subset P_F(\mathcal{B})$ for any class $\mathcal{B}$ of topological spaces.

(b) We shall occasionally make use of the following characterization of closed mappings: a mapping $f : X \to Y$ is closed iff for every $y \in Y$ and every open neighbourhood U of $f^{-1}(y)$, there exists an open neigh-

bourhood V of y such that $f^{-1}(V) \subset U$.

(c) Arhangel'skii [4] has shown that the hereditarily quotient mappings are precisely the pseudo-open mappings. Another formulation of their definition (see [24]) is that whenever $y \in \overline{A}$ in Y , then $f^{-1}(y) \cap \overline{f^{-1}(A)} \neq \phi$.

The basic implications between the various types of mappings are shown in the following diagram.



CHAPTER II

COUNTABILITY AXIOMS AND METRIZABILITY

3. Projectors.

Definitions.

(a) A space X is a Fréchet space if whenever $x \in \overline{A}$ in X, then there is a sequence in A converging to x.

(b) A space X is sequential if whenever $A \subset X$ contains the limit points of each sequence in A which converges, then A is closed.

An enlightening study of sequential and Fréchet spaces has been carried out by Franklin [14,15]. As is well known, every first countable space is a Fréchet space and every Fréchet space is sequential. A result due to Arhangel'skii [4] states that Fréchet spaces, and only these, are hereditarily quotient images of metric spaces. Franklin [14] subsequently proved that sequential spaces, and only these, are quotient images of metric spaces. He [14] also showed that every hereditarily quotient image of a Fréchet space is Fréchet, and that every quotient of a sequential space is sequential. Hence the hereditarily quotient projector of first countable spaces consists of all Fréchet spaces, while the quotient projector of first countable spaces consists of all sequential spaces. Since open maps preserve first countability, the open resolvant and the open

projector of first countable spaces are both simply the class of all first countable spaces.

Second countability is preserved by perfect maps, open maps and bi-quotient maps (see Michael [27]). Hence if F represents the class of any one of these three types of maps and $\mathcal{B}$ is the collection of all second countable spaces, then we have $R_F(\mathcal{B}) = \mathcal{B} = P_F(\mathcal{B})$.

A good deal is known about the projectors of the class M of metrizable spaces. Since every perfect image of a metrizable space is metrizable [18], we have $R_F(M) = M = P_F(M)$ for the class F of perfect maps. That the open projector of metrizable spaces is the class of first countable spaces follows from a result, obtained independently by Ponomarev [35] and Hanai [19], stating that a topological space is first countable iff it is a continuous open image of a metrizable space. For the class F of bi-quotient maps, $P_F(M)$ is the class of bi-sequential[2] spaces, as Michael [29] has recently proved that a topological space is bi-sequential iff it is a bi-quotient image of a metrizable space. The hereditarily quotient and the quotient projectors of metrizable spaces coincide with their counterparts for first countable spaces, according to the results (of Arhangel'skii and Franklin) quoted at the beginning of this section.

4. Resolvants of First Countable Spaces.

We shall frequently call upon the following simple, but useful, result. Since it provides a sufficient condition on a space X under

which every quotient mapping of X is closed, it relates to Problem (4) in the introduction.

Proposition 4.1. Every countably compact subset F of a sequential Hausdorff space X is closed.

Proof. If F is not closed, we can find a sequence (x_n) in F converging to a point $x \notin F$. Since F is countably compact, (x_n) has a cluster point $p \in F$. But then x and p cannot be separated by open sets, contradicting the assumption that X is Hausdorff, so that F must be closed.

Corollary 4.2. If X is a countably compact, sequential Hausdorff space, then every quotient mapping of X is a closed map.

Definition 4.3. Let A be a subset of a space X.

(a) A collection $\mathcal{B}$ of open subsets of X is a base at A if each member of $\mathcal{B}$ contains A, and for every open G containing A, there exists $B \in \mathcal{B}$ such that $A \subset B \subset G$.

(b) A is said to be of countable character if there exists a countable base at A.

Topological spaces in which every closed subset is of countable character have been studied by Aull [7]. In view of the following proposition, the inclusion of "regularity" in the hypothesis of many of his results is redundant. Two such results are stated in the corollary below.

Recall that a base $\mathcal{B}$ for X is point-countable iff each point of X belongs to at most countably many members of $\mathcal{B}$.

Proposition 4.4. If X is a Hausdorff space in which every closed subset is of countable character, then X is regular and acc X is countably compact.

Proof. Suppose every closed subset of X is of countable character. If X is not regular, we can find a closed subset F and a point $x \notin F$ which cannot be separated by open sets. Let $\{G_n \mid n \in N\}$ and $\{H_n \mid n \in N\}$ denote nested open bases at x and F respectively such that $x \notin H_n$ and $G_n \cap F = \phi$ for each $n \in N$. Then the sequence (x_n) formed by choosing $x_n \in G_n \cap H_n$ converges to x. Let $A = \{x_n \mid n \in N\} \cup \{x\}$. Since $X - A$ is an open neighbourhood of F, $H_m \subset X - A$ for some $m \in N$. But this implies $x_m \notin A$, a contradiction. Therefore X must be regular.

That acc X is countably compact now follows from Aull's result [7] that every regular space in which each closed subset is of countable character is the union of a countably compact set and isolated points.

Corollary 4.5. If X is a Hausdorff space whose every closed subset is of countable character, then

(a) X is perfectly normal and collectionwise normal;

(b) X is metrizable iff X has a point-countable base.

MacDonald [22] has shown the equivalence of statements (c) and (d) in the following theorem for regular X.

Theorem 4.5. The following statements are equivalent for a Hausdorff space X:

(a) every Hausdorff quotient of X is first countable;

(b) every hereditarily quotient Hausdorff image of X is first countable;

(c) every continuous closed Hausdorff image of X is first countable;

(d) every closed subset of X is of countable character.

Proof. That (a) => (b) => (c) is clear.

(c) => (d) : Suppose (c) holds and let F be a closed subset of X. Let f denote the quotient mapping obtained by identifying F to a point p, and let $\{U_n \mid n \in N\}$ be a countable base at p. Then $\{f^{-1}(U_n) \mid n \in N\}$ is a countable base at F, since f is a closed mapping.

(d) => (a) : Suppose every closed subset of X is of countable character and let f be a quotient mapping from X onto a Hausdorff space Y. We must show that each non-isolated point of Y has a countable base. Given $y \in \operatorname{acc} Y$, let $\{U_n \mid n \in N\}$ denote a countable open base at the closed set $f^{-1}(y)$. Now $\operatorname{acc} X$ is countably compact by Proposition 4.3, so that $\operatorname{acc} X - U_n$ is countably compact for each n. Since X is first countable and sequential spaces are preserved by quotient mappings, Y is sequential. Therefore $f(\operatorname{acc} X - U_n)$, being countably compact, is a

closed subset of Y, whence $X - f^{-1}(f(\text{acc } X - U_n))$ is an open neighbourhood of $f^{-1}(y)$. Set $V_n = f(U_n) \cap [Y - f(\text{acc } X - U_n)]$. Then $f^{-1}(V_n) = f^{-1}(f(U_n)) \cap [X - f^{-1}(f(\text{acc } X - U_n))]$ and is open in X, since $f^{-1}(V_n) \cap \text{acc } X \subset U_n$. Each V_n is therefore an open neighbourhood of y contained in $f(U_n)$.

To see that $\{V_n \mid n \in N\}$ forms a neighbourhood base at y, let G be a neighbourhood of y. Then $f^{-1}(y) \subset U_m \subset f^{-1}(G)$ for some $m \in N$, whence $y \in V_m \subset f(U_m) \subset G$, as required.

Corollary 4.7. If every closed subset of a Hausdorff space X is of countable character, then every quotient mapping of X is countably bi-quotient.

Proof. This follows immediately from a result of Stone [39] to the effect that a quotient mapping onto a first countable space is countably bi-quotient.

Note that, by the previous theorem, the quotient resolvant of first countable spaces is contained in the quotient resolvant of regular spaces. But MacDonald and Willard [23] have shown that the continuous resolvant of regular spaces is the class of all compact spaces, so we obtain

Corollary 4.8. Every continuous Hausdorff image of X is first countable iff X is a compact space in which every closed subset is of countable character.

The proof of the following proposition is straightforward.

Proposition 4.9. Every Hausdorff perfect image of a first countable space X is first countable iff every compact subset of X is of countable character.

Proposition 4.10. Each of the following conditions implies that every bi-quotient Hausdorff image of X is first countable:

(a) X is second countable;

(b) every closed subset of X is of countable character;

(c) X is locally compact and every compact subset of X is of countable character;

(d) X is locally countably compact and every countably compact subset of X is of countable character.

Proof. (a) Michael [27] has shown that bi-quotient mappings preserve second countability.

(b) This follows from Theorem 4.6.

(c) Every bi-quotient mapping of a locally compact space is P_2 (see Siwiec and Mancuso [38, Proposition 1.1]), and it is easily checked that every P_2 - image of a space X is first countable iff every compact subset of X is of countable character.

(d) Suppose X satisfies (d) and let f be a bi-quotient mapping from X onto a Hausdorff space Y. Given $y \in Y$, for each $x \in f^{-1}(y)$ choose a countably compact neighbourhood G_x of x. Since f is bi-quotient, there exist $G_{x_1},\cdots,G_{x_n}$ such that $y \in \operatorname{Int} f[\bigcup_{i=1}^{n} G_{x_i}] = \operatorname{Int} f(K)$, where $K = \bigcup_{i=1}^{n} G_{x_i}$ is countably compact. To see that $y \in \operatorname{Int} f(U)$ for

every neighbourhood U of the countably compact subset $K \cap f^{-1}(y)$, let $G = \text{Int } f(K) - f(K-U)$. Now $f(K-U)$, being a countably compact subset of a sequential space, is closed. Hence G is an open neighbourhood of y contained in $f(U)$. Finally, let $\{H_n \mid n \in N\}$ be a countable base at $K \cap f^{-1}(y)$. Then $\{\text{Int } f(H_n) \mid n \in N\}$ is a countable base at y.

Corollary 4.11. Every bi-quotient image of a locally compact metric space is first countable.

Since the identification of a compact subset of a Hausdorff space X yields a bi-quotient mapping of X, a necessary condition in order that every bi-quotient image of X is first countable is clearly the requirement that every compact subset of X is of countable character. (That requiring every countably compact subset to be of countable character is not necessary can be seen by considering the space of countable ordinals.) The following example shows that this condition is not sufficient even for metrizable spaces.

Example 4.12. This example, conceived by Michael [28], was presented in [38, Example 5] to show that a bi-quotient mapping of a metrizable space need not be P_2. Let X be any compact, first countable space containing a compact subset K which is not of countable character. (For example, let $X = L \times L$ where L is the lexicographically ordered square and K is the diagonal in $L \times L$.) Since X is first countable, there exists an open mapping f from a metrizable space M onto X (see [19] or [35]). Let Y be the quotient space obtained from X by identifying K to a point. Then Y is clearly not first countable and the induced mapping g is

perfect. But $g \circ f$, being the composition of two bi-quotient mappings, is a bi-quotient mapping from the metrizable space M onto Y.

5. Resolvants of Second Countable Spaces.

The following results were obtained by MacDonald [22] for regular spaces. The proofs given here, for Hausdorff spaces, are somewhat simpler.

Theorem 5.1. The following are equivalent for any Hausdorff space X:

(a) every Hausdorff quotient of X is separable metrizable;

(b) every Hausdorff quotient of X is second countable;

(c) every hereditarily quotient Hausdorff image of X is second countable;

(d) every continuous closed Hausdorff image of X is second countable;

(e) X is separable metrizable and $\operatorname{acc} X$ is compact.

Proof. That (a) => (b) => (c) => (d) is obvious.

(d) => (e) : Assuming (d), we conclude from Proposition 2.4 and Theorem 2.6 that X is a regular, hence metrizable, space and that $\operatorname{acc} X$ is compact.

(e) => (a) : This implication follows from a result originally due to Rainwater [36] and stated in Theorem 6.2, together with the fact that separability is preserved under continuous mappings.

Applying a result of Willard (see Theorem 6.1), we obtain

Corollary 5.2. The following are equivalent for any Hausdorff space X :

(a) every continuous Hausdorff image of X is second countable;

(b) every continuous Hausdorff image of X is separable metrizable;

(c) X is compact metrizable.

6. Resolvants of Metrizable Spaces.

Theorem 6.1 (Willard [43]). Every continuous Hausdorff image of X is metrizable iff X is compact metrizable.

Theorem 6.2. The following are equivalent for a metrizable space X :

(a) every Hausdorff quotient of X is metrizable;

(b) every hereditarily quotient Hausdorff image of X is metrizable;

(c) every continuous closed Hausdorff image of X is metrizable;

(d) $\operatorname{acc} X$ is compact.

Proof. That (a) => (b) => (c) is obvious. Willard [43] has shown the equivalence of (c) and (d). That (a) and (d) are equivalent was originally proved by Rainwater [36], and later by MacDonald [22] and by Alster [2].

Lemma 6.3. If f is a bi-quotient mapping of X and U is a neighbourhood of $\operatorname{acc} X$, then f restricted to U is also bi-quotient.

Proof. Let f be a bi-quotient mapping from X onto a space Y. Given $y \in f(U)$, let $G = \{G_\alpha \mid \alpha \in A\}$ be an open cover of $f^{-1}(y) \cap U$.

Then $G \cup \{x : x \in f^{-1}(y) - U\}$ is an open cover of $f^{-1}(y)$. Since f is bi-quotient, there exist $G_{\alpha 1}, \cdots, G_{\alpha n}$ and points $x_1, \cdots, x_m$ in $f^{-1}(y) - U$ such that $y \in \text{Int}\,[\bigcup_{i=1}^{n} f(G_{\alpha i}) \cup \bigcup_{i=1}^{m} f(x_i)]$. Hence $y \in \text{Int} \bigcup_{i=1}^{n} f(G_{\alpha i})$, and f restricted to U is bi-quotient.

Theorem 6.4 [CH]. If X is a 0-dimensional metrizable space, then the following are equivalent:

(a) every bi-quotient Hausdorff image of X is metrizable;

(b) every open Hausdorff image of X is metrizable;

(c) acc X is compact, or else X is locally compact and acc X is separable.

Proof. That (a) implies (b) is obvious. The equivalence of (b) and (c) was proved (on the continuum hypothesis) by Alster [2].

(c) => (a) : If acc X is compact, then every Hausdorff quotient of X is metrizable by the previous theorem. So suppose X is locally compact and acc X is separable. Let $\{U_n \mid n \in N\}$ be a cover of acc X by sets open in X and with compact closures. Then $U = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \overline{U}_n$ is locally compact and separable. By the previous lemma, f restricted to U is bi-quotient, so that $f(U)$ is second countable and regular, hence separable metrizable. But Y is the disjoint union of $f(U)$ and its discrete complement, whence Y is metrizable.

Since the 0-dimensionality of X was not used in the above proof of (c) implies (a), it is clear that either of the two conditions given in part (c) is sufficient for every bi-quotient image of a metriza-

ble space X to be metrizable. As pointed out by Alster [2], an additional sufficient condition under which every open image of a metrizable space X is metrizable is that X is the disjoint union of metrizable continua.

We conclude this section with a glimpse of what can be expected regarding resolvants in the uniform setting. Because we are digressing somewhat from the main theme, our discussion will be brief and details are omitted. To aid in this regard, familiarity with the notation, terminology and basic results in uniform spaces as given in Willard [44, Sections 35 - 41] will be assumed. All mappings are uniformly continuous, and a uniform space (X,μ) is understood to be a set X equipped with a separated covering uniformity μ. Thus a uniformity μ on X will be a filter of coverings of X with respect to star-refinement. Recall that a space (X,μ) is <u>uniformly metrizable</u> iff μ has a countable base.

As is well known, the Stone-Čech compactification of a metrizable space X is metrizable iff X is compact. More generally, it is not difficult to show that the Samuel compactification [44, p. 274] of a uniformly metrizable space (X,μ) is (uniformly) metrizable iff (X,μ) is totally bounded. Applying [44, Theorem 39.10], one then concludes that if (X,μ) is a totally bounded, uniformly metrizable space, then every uniformly continuous image of (X,μ) is uniformly metrizable. The converse follows easily, yielding a uniform analogue of Theorem 4.1:

<u>Proposition 6.5</u>. Every uniformly continuous image of a uniformly metrizable space (X,μ) is uniformly metrizable iff (X,μ) is totally bounded.

Thus, for example, every uniformly continuous image of the interval $(0,1)$ (with its usual subspace uniformity inherited from the real line) is uniformly metrizable, although there exists a non-metrizable hereditarily quotient image of $(0,1)$.

The following defintion enables us to characterize, without much effort, the F-resolvant of uniformly metrizable spaces for a large subclass F of the class of all uniform quotient maps.

Definition 6.6. Let $f : (X,\mu) \to (Y,\nu)$ be a mapping.

(a) f is a uniform quotient mapping if ν is the finest uniformity on Y which makes f uniformly continuous.

(b) f is uniformly pseudo-open if for each $U \in \mu$, there is some $V \in \nu$ such that for every $y \in Y$, $St(y,V) \subset f[St(f^{-1}y,U)]$.

It is easily checked that every uniformly pseudo-open map is a uniform quotient map, and that every uniformly open map (see Kelley [21, p. 202]) and every uniformly closed map (see Dowker [13]) is uniformly pseudo-open. In addition, every uniformly pseudo-open map is pseudo-open with respect to the induced uniform topologies. Note that this is in contrast to the general situation in which the topology induced by a quotient uniformity may be strictly coarser than the corresponding quotient topology.

Given a uniformly pseudo-open map $f : (X,\mu) \to (Y,\nu)$, there corresponds to each $U \in \mu$ a uniform cover $\{f[St(f^{-1}y,U)] \mid y \in Y\}$ of Y. It can be readily verified that the collection of all such uniform

covers of Y forms a base for ν. Therefore every uniformly pseudo-open image of a uniformly metrizable space is uniformly metrizable. In other words, if F represents either the class of uniformly open maps, uniformly closed maps or uniformly pseudo-open maps, then the F-resolvant of uniformly metrizable spaces is just the collection of all uniformly metrizable spaces.

CHAPTER III

LOCAL COMPACTNESS

7. Projectors.

The F-projectors of locally compact spaces, for the classes F of mappings of interest in this thesis, are all known. Since bi-quotient maps preserve local compactness [27, Proposition 3.4], it is clear that $R_F(L) = L = P_F(L)$ whenever F represents the class of perfect maps, open maps or bi-quotient maps and L is the collection of all locally compact spaces. Arhangel'skii [5, Theorem 3.3] has shown that every hereditarily quotient image of a locally compact space is a k'-space[3] and conversely, that every k'-space is an hereditarily quotient image of the disjoint union of all its compact subsets. Hence the hereditarily quotient projector of locally compact spaces is the class of all k'-spaces. That the quotient projector of locally compact spaces is the class of k-spaces follows from a theorem of Cohen [11]: X is a k-space iff X is a quotient of a locally compact space.

8. Resolvants of Locally Compact Spaces.

In this section we characterize the F-resolvant of locally compact spaces, where F represents either the class of continuous maps, closed maps or hereditarily quotient maps (with Hausdorff range). The continuous case can be dispensed with immediately, as a corollary to the following theorem.

Theorem 8.1 (MacDonald and Willard [23]). Every continuous Hausdorff image of X is regular iff X is compact.

Corollary 8.2. Every continuous Hausdorff image of X is locally compact iff X is compact.

For convenience, let us call a neighbourhood U of a closed subset F almost compact if U contains no non-compact closed subset disjoint from F. If X is normal, it is easy to see that whenever U is an almost compact neighbourhood of F and $F \subset V \subset U$, then V is likewise almost compact. Moreover, every compact subset of a locally compact Hausdorff space has an almost compact (in fact, compact) neighbourhood base.

The following lemma may be of some independent interest, inasmuch as it provides both an internal and an external characterization of those normal topological spaces (without isolated points) having finite remainder in their Stone-Čech compactification.

Lemma 8.3. If X is a dense-in-itself Hausdorff space, then the following are equivalent:

(a) every hereditarily quotient Hausdorff image of X is locally compact;

(b) every continuous closed Hausdorff image of X is locally compact;

(c) X is a normal, countably compact space with the property that every closed subset has an almost compact neighbourhood;

(d) X is a normal space with at most finitely many mutually disjoint closed non-compact subsets;

(e) X is normal and $|\beta X - X| < \mathcal{N}_0$.

Proof. (a) => (b) : This implication is obvious, since every continuous closed mapping is hereditarily quotient.

(b) => (c) : Suppose every continuous closed image of X is locally compact. Then X must clearly be normal, for otherwise there exist two disjoint closed subsets which cannot be separated by disjoint open sets. By identifying one of these to a point, we obtain a non-regular Hausdorff image of X under a closed mapping.

If X is not countably compact, there is an infinite closed subset $A \subset X$ with no accumulation points. Let $f : X \to Y$ be the quotient mapping obtained by identifying A to a point. It will be shown that Y is not locally compact at the point $f(A) = a$. We may assume $A = \{a_1, a_2, \cdots\}$.

Since X is locally compact and Hausdorff, it is regular. Hence we may select neighbourhoods $U_1, U_2, \cdots$ of $a_1, a_2, \cdots$ such that $\overline{U}_i \cap \overline{U}_j = \emptyset$ whenever $i \neq j$. Let $U = f(\bigcup_{i=1}^{\infty} U_i)$. Then U is a neighbourhood of a in Y and so contains a compact neighbourhood V of a. But then $f^{-1}V$ is a neighbourhood of A in X, and $f^{-1}V = V_1 \cup V_2 \cup \cdots$ where $V_i = f^{-1}V \cap U_i$. Now V_i is a neighbourhood of a_i for each i, and we can pick neighbourhoods S_i and T_i so that $a_i \in S_i$, $\overline{S}_i \subset T_i$ and $\overline{T}_i \subset V_i$ for each integer i. Now let $T = f(\bigcup_{i=1}^{\infty} T_i)$ and let $R_i = f(V_i - \overline{S}_i)$. Then T and R_i are open subsets of V. Moreover, $V = T \cup \bigcup_{i=1}^{\infty} R_i$ and no

subcover of this cover can be found, contradicting the assumption that V is compact. Thus X must be countably compact.

Finally, it is immediate that every compact subset of X has an almost compact neighbourhood. That such is also the case for any closed non-compact subset F becomes evident if one considers the closed mapping obtained by identifying F to a point.

(c) => (d) : Suppose X satisfies the properties of (c) . Then if (d) does not hold, there exists a countable collection $\{F_i \mid i \in N\}$ of mutually disjoint closed non-compact subsets of X . We shall first establish (by induction) the existence of a sequence of mutually disjoint closed subsets of X , each of which is a neighbourhood of some closed non-compact subset of X .

To this end, let $F_1' = F_1$. Choose G_1 to be an almost compact, open neighbourhood of F_1' and choose K_1 to be a closed neighbourhood of F_1' contained in G_1 . Now suppose that for each $i = 1,\cdots,n$, we have a closed, non-compact subset F_i' of F_i , a closed neighbourhood K_i of F_i' , and an almost compact, open neighbourhood G_i satisfying $F_i' \subset K_i \subset G_i$ and $K_i \cap K_j = \emptyset$ whenever $i \neq j$. Define $F_{n+1}' = F_{n+1} \cap (X - \bigcup_{i=1}^{n} G_i)$. Then F_{n+1}' is a non-empty closed, non-compact subset of X . Choose G_{n+1} to be an almost compact, open neighbourhood of F_{n+1}' whose closure misses $\bigcup_{i=1}^{n} K_i$, and choose K_{n+1} to be a closed neighbourhood of F_{n+1}' contained in G_n . We can thus inductively construct a sequence $\{K_i \mid i \in N\}$ of mutually disjoint closed subsets, each K_i being a neighbourhood of a closed non-compact subset F_i' .

To see that the existence of such a collection contradicts (c), define C to be the closure of the set of all accumulation points of sequences formed by picking exactly one point from each of infinitely many F_i'. Although C is disjoint from each F_i', every neighbourhood of C must contain all but finitely many F_i'. But this contradicts the hypothesis that C possesses an almost compact neighbourhood, thus completing the proof that (c) implies (d).

(d) <=> (e) : That (d) is equivalent to (e) follows easily from the well-known fact that, for normal X, βX is just the Wallman compactification of X. The points of $\beta X - X$, then, are in one-to-one correspondence with the free closed ultrafilters on X.

(d) => (a) : Suppose X satisfies (d) and possesses at most n mutually disjoint closed non-compact subsets. Let f be an hereditarily quotient mapping of X onto a space Y. Given an arbitrary point $y \in Y$, we must show that y has a compact neighbourhood. Now if $f^{-1}y$ is compact, then $f^{-1}y$ has a compact neighbourhood K, whence $f(K)$ is a compact neighbourhood of y. So suppose $f^{-1}y$ is a closed non-compact subset of X. Then $f^{-1}y$ possesses a closed, almost compact neighbourhood V. For otherwise, normality allows us to choose a collection of $n+1$ mutually disjoint closed non-compact subsets, contrary to hypothesis.

Now since $f(V)$ is a neighbourhood of y, the proof will be complete once we show that $f(V)$ is compact. But this follows easily, since if H is any open neighbourhood of y, then $V - f^{-1}(H)$ is compact, whence $f(V) - H$ is compact. Therefore $f(V)$ is compact.

Remarks 8.4. In the proof of the previous lemma, the hypothesis that X is dense-in-itself was used only to show that (b) implies X is countably compact. It is easily checked that the same proof justifies the following statements for any Hausdorff space X :

(i) If X satisfies Lemma 8.3 (b), then acc X is countably compact and every closed subset of X has an almost compact neighbourhood.

(ii) In Lemma 8.3, (c) => (d) <=> (e) => a => (b) . That (d) => (c) for any Hausdorff space X can also be verified easily, for any non-countably compact space contains a sequence (x_n) having no cluster point. But (x_n) can then be written as the union of countably many mutually disjoint closed non-compact subsets.

Thus we have (c) <=> (d) <=> (e) for any Hausdorff space X . The following example shows that "normality" is not a redundant hypothesis in statement (d).

Example 8.5. A non-regular, countably compact Hausdorff space X with the property that of every pair of disjoint closed subsets of X , at least one is compact.

Let $X = [0,\omega_1) \cup \{\omega_1\}$, with the space of countable ordinals carrying the usually order topology and basic neighbourhoods of ω_1 being of the form $[\alpha,\omega_1] - L$ where $\alpha \in [0,\omega_1)$ and L is the set of limit ordinals in $[0,\omega_1)$. Then $\{\omega_1\}$ can't be separated from the closed set L , and every closed non-compact subset of X contains a residual subset of L .

Lemma 8.6. If X is normal and $X = X_o \oplus D$, where D consists of points isolated in X, then

(i) if every continuous closed Hausdorff image of X is locally compact, the same is true for X_o;

(ii) the restriction to X_o of an hereditarily quotient mapping on X is hereditarily quotient.

Proof. (i) Given a continuous closed mapping f on X_o, define $g : X \to f(X_o) \oplus D$ by $g\big|_{X_o} = f$ and $g\big|_D$ is the identity map. Since g is a continuous closed mapping on each component, it is a continuous closed mapping on X. Therefore $f(X_o) \oplus D$, and hence $f(X_o)$, is locally compact.

(ii) Let f be an hereditarily quotient mapping from X onto a space Y. To see that the restriction of f to X_o is hereditarily quotient, choose a point $y \in f(X_o)$ and an open neighbourhood U of $f^{-1}(y) \cap X_o$ in X_o. Since U is also open in X, $U \cup (f^{-1}(y) \cap D)$ is an open neighbourhood of $f^{-1}(y)$ in X. Therefore $f[U \cap (f^{-1}(y) \cup D)] = f(U)$ is a neighbourhood of y in Y, hence also in $f(X_o)$.

Theorem 8.7. The following statements are equivalent for any Hausdorff space X:

(a) every hereditarily quotient Hausdorff image of X is locally compact;

(b) every continuous closed Hausdorff image of X is locally compact;

(c) X is normal and $X = X_0 \oplus D$, where $|\beta X_0 - X_0| < \aleph_0$ and D consists of points isolated in X.

Proof. (a) => (b) is obvious.

(b) => (c) : Suppose every continuous closed Hausdorff image of X is locally compact. Then by 8.4 (i), acc X is countably compact and has an almost compact neighbourhood X_0. Easily, X_0 is also countably compact. Since X_0 is clopen, it follows from 8.4 (i) and 8.6 (i) that X_0 satisfies property (c) of Lemma 8.3, so that $|\beta X_0 - X_0| < \aleph_0$. Thus $X = X_0 \oplus (X - X_0)$, where $|\beta X_0 - X_0| < \aleph_0$ and $(X - X_0)$ consists of points isolated in X. That X must be normal follows as before.

(c) => (a) : Suppose X satisfies (c) and let f be an hereditarily quotient mapping of X onto a space Y. By 8.6 (ii), the restriction of f to X_0 is hereditarily quotient. Hence $f(X_0)$ is locally compact, by 8.4 (ii), so that Y is the disjoint union of a locally compact space and a discrete space. Therefore Y is locally compact, and we are done.

Example 8.8. Given any positive integer n, the disjoint union of n copies of the countable ordinals provides an example of a normal space X with $|\beta X - X| = n$, and hence an example of a space whose every hereditarily quotient Hausdorff image is locally compact.

The Tychonoff plank shows that "normality" cannot be omitted from statement (c) in the previous theorem.

Example 8.9. Although the closed and hereditarily quotient resolvants of locally compact spaces coincide, the following example illustrates that closed and hereditarily quotient mappings need not coincide on spaces belonging to these resolvants.

Let $X = \Omega \oplus L$, where L is the space of limit ordinals in $[0,\omega_1)$, and let f be the obvious map from X onto Ω. Then X is a normal space with $|\beta X - X| = 1$. But although the map f is bi-quotient, and hence hereditarily quotient, it is neither open nor closed.

That condition (c) of Theorem 8.7 is necessary for every Hausdorff quotient of X to be locally compact is clear, but sufficiency remains unproved. It follows from the next corollary, however, that every Hausdorff quotient of the space X described in Example 8.8 is locally compact.

Recall that a mapping $f : X \to Y$ is a compact mapping iff $f^{-1}(y)$ is compact for every $y \in Y$.

Proposition 8.10. (a) Every Hausdorff quotient of X is locally compact iff X is normal, $X = X_o \oplus D$ where $|\beta X_o - X_o| < \mathcal{N}_o$ and D consists of points isolated in X, and every Hausdorff quotient of X_o is regular.

(b) If X is a normal space with $|\beta X - X| = 1$ and f is a quotient mapping of X onto a non-regular Hausdorff space Y, then f is a compact mapping.

Proof. (a) Necessity follows from Theorem 8.7. To show sufficiency, it is enough to prove that every Hausdorff quotient of X_o is locally compact. Now given any quotient mapping $f : X_o \to Y$, Y is normal by hypothesis.

Let $f^{\beta} : \beta X_o \to \beta Y$ be the Stone extension mapping. Then $f^{\beta}(\beta X_o - X_o) = (\beta Y - Y)$ is a closed (in fact, finite) subset of βY, whence Y is locally compact.

(b) Suppose f is a non-compact mapping and choose $y_o \in Y$ such that $f^{-1}(y_o)$ is not compact. If F is a closed subset of Y which cannot be separated from a point $p \notin F$, then F is non-compact. Hence $f^{-1}(F)$ is a non-compact subset of X and so must contain $f^{-1}(y_o)$, for otherwise we obtain two disjoint, closed non-compact subsets of X, contradicting the hypothesis that $|\beta X - X| = 1$.

Since Y is Hausdorff, there exists a neighbourhood V of y_o such that $p \notin \overline{V}$. Now $F \cap (Y-V)$ is a closed subset of Y, so that $f^{-1}[F \cap (Y-V)]$ is a closed subset of X missing $f^{-1}(y_o)$. Therefore, $f^{-1}[F \cap (Y-V)]$, hence also $F \cap (Y-V)$, is compact and we can find a neighbourhood W of $F \cap (Y-V)$ such that $p \notin \overline{W}$. Thus $V \cup W$ is a neighbourhood of F whose closure misses p, a contradiction, so that f must be a compact mapping.

Corollary 8.11. Every Hausdorff quotient of a sequential space X is locally compact iff X is normal and $X = X_o \oplus D$, where $|\beta X_o - X_o| < \mathfrak{N}_o$ and D consists of points isolated in X.

Proof. Necessity is immediate. To prove sufficiency, let f be any quotient mapping of X onto a space Y. Then f restricted to X_o is a closed map by Corollary 4.2, whence Y is the disjoint union of the normal space $f(X_o)$ and its discrete complement. Therefore Y is normal and hence, by part (a) of the proposition, also locally compact.

Problem 8.12. If X is a normal space with $|\beta X - X| < \aleph_0$, is every Hausdorff quotient of X regular? An affirmative answer will imply that every Hausdorff quotient of X is also locally compact.

CHAPTER IV

PARACOMPACTNESS

9. Introduction.

The F-projectors (for each of five classes F of quotient mappings) of both the collection of paracompact M-spaces[4] and the collection of locally compact paracompact spaces have been documented by Michael [29]. A typical result in this area states that the class of all locally compact spaces is both the open and the bi-quotient projector of locally compact paracompact spaces. This follows from a result of Morita [31, Theorem 2]: every locally compact space is an open image of a locally compact paracompact space. For the class of paracompact M-spaces, Wicke [42, Theorem 2] has shown the open projector to be the class of spaces of point-countable type.[5]

Since closed mappings preserve paracompactness (Michael [26]), it is clear that $R_C(\mathcal{B}) = \mathcal{B} = P_C(\mathcal{B})$ whenever $\mathcal{B}$ is the collection of all paracompact spaces and C is either the class of closed maps or perfect maps. Aull [8, Theorem 2] has established that every continuous Hausdorff image of a regular space X is countably paracompact iff X is countably compact. It then follows easily that the continuous resolvant of paracompact spaces is the class of all compact spaces, a result due to MacDonald [22]. For the class F of all continuous mappings with regular range, MacDonald and Willard [23, Theorem 3.2] have shown the F-resolvant of paracompact spaces to be the class of regular Lindelöf spaces. Aull

[8, Theorem 1] has recently extended this result from the class of paracompact spaces to the class of all normal spaces with property P, where P is any property of a space X which is closed-hereditary[6] and which, together with the condition that every uncountable subset of X has a cluster point, implies that X is Lindelöf. Thus, for example, the F-resolvant of normal meta-Lindelöf[7] spaces is the class of regular Lindelöf spaces.

The problem of characterizing the quotient and the open resolvants of paracompact spaces appears to be difficult. Some partial solutions to the problem are provided in the next two sections.

10. Quotient Resolvants of Paracompact Spaces.

Definition 10.1. A covering A of a space X will be called compatible with the topology of X (or simply, compatible) if a subset K of X is closed whenever $K \cap A$ is closed for every $A \in A$.

Defintion 10.2 [30]. A space X has the weak topology with respect to a covering A if it is a closed, compatible, closure-preserving cover of X.

Note that, in particular, a topological space X always has the weak topology with respect to any locally finite closed cover of X.

Theorem 10.3 (Morita [30]). If X has the weak topology with respect to a covering $\{A_\alpha\}$, then X is paracompact iff each subspace A_α is paracompact.

Corollary 10.4 [30]. If $\{A_n \mid n \in N\}$ is a countable compatible covering of a topological space X, then X is normal (respectively paracompact) if each A_n is normal (respectively paracompact).

Some progress towards characterizing the quotient resolvant of paracompact spaces is given in the following two theorems.

Theorem 10.5 (Morita [31]). If X has a countable compatible covering consisting of compact subsets, then every Hausdorff quotient of X is paracompact.

Corollary 10.6 [31]. If X is a locally compact, Lindelöf space, then every Hausdorff quotient of X is paracompact.

Theorem 10.7 (MacDonald and Willard [23]). (a) If X is regular and acc X is Lindelöf, then every regular quotient of X is paracompact.

(b) If X is paracompact and each point of a dense subset of acc X has a countable neighbourhood base, then every regular quotient of X is paracompact iff acc X is Lindelöf.

In view of the above theorem, MacDonald and Willard [23] conjectured that every regular quotient of X is paracompact only if acc X is Lindelöf. Example 10.11 below, however, shows that such is not the case even for quotient mappings with Hausdorff range. But before giving the counterexample, we need some additional terminology and a theorem.

The proof of the following lemma is contained in the proof of [31, Lemma 3].

Lemma 10.8. If $\mathcal{A}$ is a compatible cover of X and f is a quotient mapping of X onto Y with $f(A)$ closed for each $A \in \mathcal{A}$, then $\{f(A) \mid A \in \mathcal{A}\}$ is a compatible cover of Y.

For the remainder of this section, n and m will always be infinite cardinals and $W(n)$ will denote the least ordinal of cardinality n.

Recall that a space X is a P-space [16, 4J] iff every G_δ in X is open. A space X is n-discrete [20, p. 135] iff each intersection of n or fewer open subsets is open, and $<n$ - discrete [32, p. 386] if it is m-discrete for each $m < n$. Thus an $\aleph_0$ - discrete space is just a P-space.

Noble [32, p. 387] has called a space n-compact iff every open cover admits a subcover of cardinality less than n. Thus an $\aleph_0$ - compact space is simply a compact space, while an $\aleph_1$ - compact space is just a Lindelöf space.

Our next result generalizes Theorem 10.5.

Theorem 10.9. If X is a paracompact $<n$ - discrete space having a compatible cover $\{A_\alpha \mid \alpha < W(n)\}$ consisting of n-compact subsets, then every Hausdorff quotient of X is paracompact.

Proof. Let f be any quotient mapping of X onto a Hausdorff space Y. It is easily verified that $<n$ - discrete spaces are preserved by quotient mappings, so that Y is also $<n$ - discrete. To see that every n - compact subset B of Y is closed, choose any point $p \notin B$. Now each point $y \in B$ has an open neighbourhood U_y such that $p \notin \overline{U}_y$. Since

$\{U_y \mid y \in B\}$ covers B and B is n-compact, there exists a subcover $\{U_{y(\alpha)} \mid \alpha \in \Lambda\}$ where $|\Lambda| < n$. But Y is $<n$-discrete, so that $p \notin \overline{\bigcup_{\alpha\in\Lambda} U_{y(\alpha)}} = \bigcup_{\alpha\in\Lambda} \overline{U}_{y(\alpha)}$. Hence $Y - \overline{\bigcup_{\alpha\in\Lambda} U_{y(\alpha)}}$ is an open neighbourhood of p whose closure misses B. Therefore every n-compact subset of Y is closed.

Note that every closed subset and every continuous image of an n-compact space is n-compact. Thus $f(A_\alpha)$ is closed in Y for each $\alpha < W(n)$. Moreover, although f need not be a closed mapping, its restriction to each A_α is closed.

Since paracompactness is preserved under closed mappings, $f(A_\alpha)$ is paracompact for each $\alpha < W(n)$. Hence $\{f(A_\alpha) \mid \alpha < W(n)\}$ is a closed compatible cover of Y by paracompact subspaces, applying Lemma 10.8. Now define $K_\beta = \bigcup_{\alpha<\beta} f(A_\alpha)$ for each $\beta < W(n)$. Note that each K_β is closed (since every union of $<n$ closed subsets is closed) and paracompact (by Theorem 10.3). Also, it is easily seen that $\{K_\beta : \beta < W(n)\}$ is a compatible cover of Y. Hence $\{K_\beta : \beta < W(n)\}$ is a closure-preserving, closed compatible cover of Y by paracompact subspaces. Applying Theorem 10.3, we conclude that Y is paracompact.

Corollary 10.10. If X is a paracompact P-space having a compatible cover $\{A_\alpha \mid \alpha<\omega_1\}$ consisting of Lindelöf subsets, then every Hausdorff quotient of X is paracompact.

Example 10.11. A paracompact space X whose every Hausdorff quotient is paracompact, although $\operatorname{scc} X$ is not Lindelöf.

Let Y denote the set of ordinals less than or equal to the first uncountable ordinal ω_1, equipped with the topology rendering each point isolated except ω_1, whose neighbourhoods consist of all cocountable subsets of Y. Define X to be the disjoint union of uncountably many copies of Y. Then X, being the disjoint union of P-spaces, is a P-space. Although acc X is clearly non-Lindelöf, every Hausdorff quotient of X is paracompact by the previous corollary.

Incidentally, this example also shows that the quotient resolvant of paracompact spaces is not contained in this class of all k-spaces.

Example 10.12. Although every (completely regular) P-space is 0-dimensional, the hypothesis that X is a P-space in Corollary 10.10 cannot be replaced by the requirement that X is 0-dimensional, even if "Lindelöf" is strengthened to "compact". For if X is the disjoint union of uncountably many convergent sequences, then X has a non-paracompact Hausdorff quotient, as will be seen in Corollary 10.21.

After first establishing some preliminary lemmas, we shall conclude this section with a characterization of those paracompact spaces of point-countable type belonging to the quotient resolvant of paracompact spaces. The only sufficient conditions hitherto available for non-0-dimensional spaces (viz. Theorems 10.5 and 10.7) imply that X is Lindelöf. That X need not be Lindelöf is apparent from the following lemma which slightly improves Corollary 10.6.

Lemma 10.13. If X is locally compact and acc X is Lindelöf, then every Hausdorff quotient of X is paracompact.

Proof. Let f be a quotient mapping of X onto a Hausdorff space Y and let $\{U_i \mid i \in N\}$ be a countable open (in X) cover of acc X such that $\overline{U}_i$ is compact for each i, where $\overline{U}_i$ denotes the closure of U_i in X. Then $U = \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} \overline{U}_i$ is an open, hence locally compact, subspace of X containing acc X. Since $\bigcup_{i=1}^{\infty} \overline{U}_i$ is also Lindelöf, U is paracompact. Noting that X is the disjoint union of the clopen set U and its discrete complement, we see that f restricted to U remains a quotient map, and $f(U)$ is paracompact by Corollary 10.6. That Y is paracompact now follows easily from the fact that $Y - f(U)$ is a closed set consisting of isolated points.

Recall that a space X is said to be of point-countable type [5] iff X is covered by a collection of compact subspaces of countable character. If $x \in X$, we shall say that X is of point-countable type at x whenever x belongs to a compact subspace of countable character. Every first countable space is clearly of point-countable type. Arhangel'skii [5, Theorem 3.8] has shown that every topologically complete space X (i.e. X is a G_δ is βX [10]), and hence every locally compact space, is of point-countable type.

Lemma 10.14. If a space X is of point-countable type at $x \in$ acc X and U is an open neighbourhood of x, then there exists a compact set K of countable character such that $x \in K \subset U$ and such that some sequence in $U - K$ is eventually in every neighbourhood of K.

Proof. The first part of the lemma is essentially a restatement of a result due to Wicke [42, Lemma 2] stating that a Hausdorff space X is

of point-countable type iff whenever U is open in X and $x \in U$, there exists a compact set K of countable character such that $x \in K \subset U$.

The second part of the conclusion is immediate if $K \not\subset U$. So suppose $x \in K = U$. Then we can find a sequence (U_i) of open neighbourhoods of x such that $U_1 = K$ and $\overline{U}_{i+1} \not\subset U_i$ for each $i \in N$. Let $K' = \bigcap_{i=1}^{\infty} \overline{U}_i = \bigcap_{i=1}^{\infty} U_i$. Then K' is a (non-open) compact G_δ subset of K and so is of countable character. If we choose $x_i \in U_i - \overline{U_{i+1}}$ for each $i \in N$, then (x_i) is a sequence in $U - K'$ which is eventually in every neighbourhood of K'.

Remark 10.15. Suppose x is a non-isolated point in X and x belongs to a compact subspace of countable character. It is not necessarily true that some point of K is the limit of a sequence in $X - K$, as the following example illustrates.

Let $X = \beta N$, the Stone-Čech compactification of the natural numbers, and let $K = \beta N - N$. Then K is a compact subspace of countable character, but no point of K is the limit of a sequence in N.

The following two lemmas improve [23, Theorems 4.3 and 2.2] respectively, employing essentially the same techniques.

Lemma 10.16 [CH]. If X is paracompact and of point-countable type at a dense set of points of $\operatorname{acc} X$, then every Hausdorff quotient of X is regular only if $\operatorname{acc} X$ is Lindelöf.

Proof. Suppose that X is paracompact and of point-countable type at a dense set of points of acc X, but that acc X is not Lindelöf. Then by [23, Lemma 3.1], acc X contains an uncountable subset $\{x_\alpha \mid \alpha < \omega_1\}$ having no accumulation point. Since X is collectionwise normal, there exists a discrete family $\{U_\alpha \mid \alpha < \omega_1\}$ of open sets such that $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$. The hypothesis that X is of point-countable type at a dense set of points of acc X and Lemma 10.14 together imply the existence of a collection $\{K_\alpha \mid \alpha < \omega_1\}$ of (non-open) compact subsets of countable character such that $K_\alpha \subset U_\alpha$ for each $\alpha < \omega_1$.

Let Y be the quotient space obtained from X by identifying each K_α to a point, say y_α, and note that each $y_\alpha \in$ acc Y. Now $\{f(U_\alpha) \mid \alpha < \omega_1\}$ is a discrete collection of open sets with $y_\alpha \in f(U_\alpha)$, and each y_α has a countable base. Hence we can find a collection $\{(y_{\alpha n})_{n=1}^{\infty} \mid \alpha < \omega_1\}$ of disjoint sequences of distinct points such that $(y_{\alpha n}) \to y_\alpha$ for each α. Let $A = \bigcup_{\alpha<\omega_1} (\{y_\alpha\} \cup \bigcup_{n=1}^{\infty} \{y_{\alpha n}\})$. Now given such a closed subset A of a paracompact space Y, MacDonald and Willard [23, p. 1085] have constructed (on the continuum hypothesis) a non-normal regular quotient of Y. But this in turn will yield a non-regular Hausdorff quotient of Y, and hence of X, by identifying to a point one of two disjoint closed subsets which can't be separated. That completes the proof of the lemma.

Lemma 10.17. If X is of point-countable type and every Hausdorff quotient of X is regular, then either

(a) acc X is countably compact, or

(b) X is locally countably compact.

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Proof. If neither (a) nor (b) holds, we can find in acc X a point y having no countably compact neighbourhood and a sequence (x_n) in acc X - {y} having no cluster point. Let U be a closed neighbourhood of y containing no x_n. By the regularity of X, we can inductively choose a sequence (V_n) of pairwise disjoint open sets such that $x_n \in V_n \subset X - U$ for each n. According to Lemma 10.14, there exist compact subsets $K, K_1, K_2, \cdots$ of countable character such that $y \in K \subset \text{Int } U$ and $x_n \in K_n \subset V_n$. For each $n \in N$, choose a sequence $(x_{ni} \mid i \in N)$ in $V_n - K_n$ which is eventually in every neighbourhood of K_n. Let $\{U_n \mid n \in N\}$ be a nested neighbourhood base at K with $U_1 = U$ and such that for each $n \in N$, $U_n - U_{n+1}$ contains a sequence $(y_{ni} \mid i \in N)$ having no cluster point.

Now identify K to the point y and identify x_{ni} with y_{ni} for all $n,i \in N$. The resulting (Hausdorff) quotient is non-regular, since the closed set $\overline{\cup\{K_n \mid n \in N\}}$ cannot be separated from the point y. But this contradicts our assumption, so that either (a) or (b) must hold.

Remark 10.18. The requirement that X be of point-countable type cannot be omitted from the hypothesis of the previous lemma, for consider the quotient space X obtained from the real line by identifying the integers. Clearly, X is not of point-countable type and satisfies neither (a) nor (b) of Lemma 10.17. But every Hausdorff quotient of the real line, and hence of X, is regular by Corollary 10.6.

Theorem 10.19. Let X be a paracompact space of point-countable type. Then the following are equivalent:

(a) every Hausdorff quotient of X is paracompact;

(b) every Hausdorff quotient of X is regular;

(c) acc X is compact, or else X is locally compact and acc X is Lindelöf.

Proof. (a) => (b) : This is obvious.

(b) => (c) : This implication follows from Lemmas 10.16 and 10.17 combined with the fact that a countably compact, paracompact space is compact.

(c) => (a) : If acc X is compact, then every Hausdorff quotient is regular [23, Theorem 2.1] and, because acc X is Lindelöf, every regular quotient of X is paracompact. If acc X is Lindelöf and X is locally compact, then every Hausdorff quotient of X is paracompact by Lemma 10.13.

Corollary 10.20. If X is a metric space, then every Hausdorff quotient of X is paracompact (equivalently, regular) iff acc X is compact, or else X is locally compact and acc X is separable.

Corollary 10.21. If X is a locally compact, paracompact space, then every Hausdorff quotient of X is paracompact (equivalently, regular) iff acc X if Lindelöf.

Remarks 10.22. (1) The independence of the two parts of condition (c) of Theorem 10.19 is easily checked. A convergent sequence with each non-

limit point expanded to an infinite discrete set satisfies the first part but not the second, while the real line works the other way.

(ii) Theorem 10.19 suggests that the quotient resolvants of regular spaces and of paracompact spaces coincide. However, the space of countable ordinals together with Corollary 8.11 forms a counterexample.

Problem 10.23. Does the class of paracompact spaces belonging to the quotient resolvant of regular spaces coincide with the quotient resolvant of paracompact spaces?

11. Open and Bi-Quotient Resolvants.

Any condition given in Section 10 which is sufficient for a space X to belong to the quotient resolvant of paracompact spaces is clearly also sufficient for X to belong to both the open and the bi-quotient resolvants of paracompact spaces. Additional results concerning the open and the bi-quotient resolvants of paracompact spaces are provided in this section.

Lemma 11.1. If X is a locally countably compact, sequential, normal space, then every bi-quotient Hausdorff image of X is regular.

Proof. Let f be a bi-quotient mapping from X onto a Hausdorff space Y. It suffices to show that every point $y \in Y$ has a closed neighbourhood which is a normal subspace of Y. Given $y \in Y$, let $\{U_x \mid x \in f^{-1}(y)\}$ be a cover of $f^{-1}(y)$ by countably compact subsets

such that $x \in \text{Int } U_x$ for each $x \in f^{-1}(y)$. Then there exist $U_{x1}, \cdots, U_{xn}$ such that $y \in \text{Int } f(U)$, where $U = \bigcup_{i=1}^{n} U_{xi}$. Using Proposition 4.1 together with the fact that every closed subset of a sequential space is sequential, we see that f restricted to the normal subspace U is a closed mapping. Therefore the closed neighbourhood $f(U)$ of y is a normal subspace of Y.

Lemma 11.2. If X is a Hausdorff space of point-countable type, then either of the following conditions implies that X is regular:

(a) acc X is countably compact;

(b) X is a locally countably compact, sequential space.

Proof. (a) Given any point $x \in \text{acc } X$ and a closed set F not containing x, there exists a compact set K of countable character such that $x \in K \subset X - (F \cap \text{acc } X)$, by Lemma 10.14. If $\{G_i \mid i \in N\}$ is a nested open base at K, then $\{X - \overline{G}_i \mid i \in N\}$ is an open cover of $F \cap \text{acc } X$. Since $F \cap \text{acc } X$ is countably compact, $x \in G_n \subset \overline{G}_n \subset X - (F \cap \text{acc } X)$ for some $n \in N$. But then $U = G_n \cap (X-F)$ is an open neighbourhood of x whose closure misses F. Therefore X is regular.

(b) Let X be a locally countably compact, sequential Hausdorff space of point-countable type. If X is not regular, then there exists a closed set F and a point $x \notin F$ which cannot be separated. Let U be a (closed) countably compact neighbourhood of x. Then by Lemma 10.14, there exists a compact set K of countable character such that $x \in K \subset \text{Int } U \cap (X-F)$. Let $\{G_n \mid n \in N\}$ be a countable nested base at K such that $G_1 \subset \text{Int } U \cap (X-F)$, and choose a point $x_n \in \overline{G}_n \cap F$ for each n. Now

the sequence (x_n), being contained in U, must have a cluster point $p \notin K$. But then the compact set K cannot be separated from the point p, a contradiction. Therefore X must be regular.

If $f : X \to Y$ is an open mapping, then $\operatorname{acc} Y \subset f(\operatorname{acc} X)$ and Y is of point-countable type whenever X is. Thus we obtain

Corollary 11.3. If X is a Hausdorff space of point-countable type satisfying either (a) or (b) of Lemma 11.2, then every open Hausdorff image of X is regular.

Example 11.4. The hypothesis that X be locally countably compact cannot be dropped from Lemma 11.2 (b) as shown by the following well-known example of a non-regular Hausdorff space.

Let X be the real line with each point except 0 having its usual neighbourhood system, and let neighbourhoods of 0 be of the form $U - A$ where $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then X is first countable, hence sequential and of point countable type, but no neighbourhood of 0 is countably compact.

The proof of the following lemma uses the techniques of the proof of Lemma 10.17 and of Alster's proof [2] of the equivalence of (b) and (c) in Theorem 6.4.[6] Recall that every 0-dimensional space is (completely) regular.

Lemma 11.5. If X is a 0-dimensional space of point-countable type and every open image of X is regular, then either

(a) acc X is countably compact, or

(b) X is locally countably compact.

Proof. If neither (a) nor (b) holds, we can find in acc X a point x_o having no countably compact neighbourhood and a sequence (x_n) in acc X - $\{x_o\}$ having no cluster point. Let $\{U_n \mid n = 0,1,2,\cdots\}$ be a collection of pairwise disjoint clopen subsets whose union is X and let $\{K_n \mid n = 0,1,2,\cdots\}$ be a collection of compact subsets of countable character such that $x_n \in K_n \subset U_n$ for $n = 0,1,2,\cdots$. Then for each n, we can choose a base $\{U_{nm} \mid m \in N\}$ at K_n consisting of clopen sets such that $U_{n\,m+1} \subsetneq U_{nm}$ for each $m \in N$ and such that $U_{om} - U_{o\,m+1}$ is not countably compact for each $m \in N$. But this last statement implies the existence, for each $m \in N$, of a cover $\{V_{mi} \mid i \in N\}$ of $U_{om} - U_{o\,m+1}$ by disjoint clopen subsets.

Consider the quotient space Y obtained from X by identifying $V_{mn} \cup (U_{nm} - U_{n\,m+1})$ to a point for each $n,m \in N$. If f is the induced quotient map, then f is an open map onto the Hausdorff space Y. But since the compact set K_o cannot be separated in Y from $\overline{\bigcup_{n=1}^{\infty} K_n} = \bigcup_{n=1}^{\infty} K_n$, Y is not regular, a contradiction. Therefore either (a) or (b) must hold.

Combining Lemma 11.5 with Corollary 11.3, we obtain

Theorem 11.6. If X is a 0-dimensional, sequential space of point-countable type, then every open image of X is regular iff acc X is countably compact or else X is locally countably compact.

Corollary 11.7. If X is a 0-dimensional, first countable space, then every open image of X is regular iff acc X is countably compact or

else X is locally countably compact.

The proof of our next theorem makes use of the space ψ of Isbell which is described in [16,5I] and which can be constructed as follows. Let E be an infinite collection of infinite subsets of N, maximal with respect to the property that if E_1 and E_2 are distinct members of E, then $E_1 \cap E_2$ is finite. If $(y_{\alpha n} \mid n \in N)$ denotes a member of E, we may (on the continuum hypothesis) write $E = \cup\{(y_{\alpha n} \mid \alpha < \omega_1)\}$. Now choose a set $D = \{y_\alpha \mid \alpha < \omega_1\}$ of distinct points not belonging to N and define $\psi = N \cup D$ with the following topology: each point of N is isolated, while a neighbourhood of y_α is any set containing y_α and all but finitely many points of $(y_{\alpha n})$. Then ψ is a non-normal, locally compact, first countable Hausdorff space.

Theorem 11.8 [CH]. If X is a 0-dimensional, paracompact space of point-countable type, then the following statements are equivalent:

(a) every bi-quotient image of X is paracompact;

(b) every bi-quotient image of X is normal;

(c) every open image of X is paracompact;

(d) every open image of X is normal;

(e) $\operatorname{acc} X$ is compact, or else X is locally compact and $\operatorname{acc} X$ is Lindelöf.

Proof. That (a) implies each of (b) and (c), and that each of (b) and (c) implies (d), are obvious. The implication (e) => (a) follows from Lemma 10.13 combined with the fact that a Hausdorff space X is paracompact whenever $\operatorname{acc} X$ is compact.

(d) => (e) : Suppose every open image of X is normal. Then by Lemma 11.5 and the fact that a countably compact, paracompact space is compact, either $\operatorname{acc} X$ is compact or X is locally compact. If $\operatorname{acc} X$ is not Lindelöf, then $\operatorname{acc} X$ contains an uncountable discrete subset $\{x_\alpha \mid \alpha < \omega_1\}$. By collectionwise normality and Lemma 10.14, there exists a collection $\{K_\alpha \mid \alpha < \omega_1\}$ of compact subsets of countable character and a discrete collection $\{U_\alpha \mid \alpha < \omega_1\}$ of mutually disjoint clopen sets whose union is X such that $x_\alpha \in K_\alpha \subset U_\alpha$. For each $\alpha < \omega_1$, let $\{U_{\alpha n} \mid n \in N\}$ be a base at K_α consisting of clopen sets such that $U_{\alpha n+1} \subsetneq U_{\alpha n}$.

Now let $E = \cup\{(y_{\alpha n}) \mid \alpha < \omega_1\}$ be the collection defined above in the description of ψ, and define a mapping $f : X \to Y = N \cup \bigcup_{\alpha<\omega_1} K_\alpha$ as follows: f restricted to each K_α is the identity map and $f(U_{\alpha n} - U_{\alpha n+1}) = y_{\alpha n}$ for each $n \in N$, $\alpha < \omega_1$. If Y is assigned the quotient topology, then f is an open mapping from X onto Y. Note that Y can be mapped onto ψ by a closed mapping, by identifying each K_α with y_α. Hence Y is not normal, for if it were then ψ would be normal, which it is not. Since Y being a non-normal open image of X contradicts our hypothesis, we conclude that $\operatorname{acc} X$ must be Lindelöf.

Remark 11.9. The idea for the mapping f in the previous proof comes from the proof of [23, Theorem 4.3], where a similar mapping is defined onto ψ. In their case, however, the mapping need not be open.

The previous theorem suggests the following two problems, similar to one posed by Alexandroff (see [6, Problems 1.2 and 1.3]): is every normal space that is an open compact image of a metric space metrizable?

Problems 11.10. (i) Does the class of paracompact spaces belonging to the open resolvant of normal spaces coincide with the open resolvant of paracompact spaces?

(ii) Does the class of metrizable spaces belonging to the open resolvant of paracompact spaces coincide with the open resolvant of metrizable spaces?

An affirmative answer of Problem 11.10 (ii) is obtained in the case of metrizable spaces which are locally separable, since every continuous open image of a locally separable, metrizable space is locally metrizable and a locally metrizable paracompact space is metrizable.

Remark 11.11. Resolvants behave badly as far as subspaces and extended topologies are concerned. For example, every Hausdorff quotient of the real line is paracompact, by Corollary 10.6. But by Corollary 11.7, there exists a non-regular Hausdorff open image of both the subspace of rationals and the Sorgenfrey line.

CHAPTER V

REGULARITY AND NORMALITY

12. Introduction.

The F-resolvant of the class of regular spaces will be of interest whenever one attempts to characterize the F-resolvant of a class of spaces which are regular (for example, locally compact or paracompact spaces), and conversely, any attempt to characterize the latter will shed some light on the former. This being the case, we now have at our disposal several results, established in earlier chapters, concerning the F-resolvants of regular spaces. The present chapter will, for the most part, simply serve as a compendium of these results.

Since regularity is preserved by perfect mappings and normality by closed mappings, the perfect resolvant of regular spaces is clearly the class of all regular spaces, while the perfect and the closed resolvants of normal spaces are both simply the class of all normal spaces. It is easily seen that the F-resolvant of regular spaces coincides with the F-resolvant of normal spaces whenever F is the class of closed maps, hereditarily quotient maps, quotient maps or continuous maps. That such is not the case for the class F of bi-quotient maps follows from the facts that there exist non-normal locally compact spaces and every bi-quotient image of a locally compact space is locally compact. Observe also that $X = X_0 \oplus D$, where D consists of points isolated in X, belongs to the F-resolvant of regular (normal) spaces whenever X_0 does.

13. Resolvants of Regular and Normal Spaces.

Quotient resolvants of regular spaces were first isolated for study by MacDonald and Willard [23], who proved both the following theorem and Proposition 13.2 (i).

Theorem 13.1 [23]. Every continuous Hausdorff image of X is regular iff X is compact.

Proposition 13.2. Every Hausdorff quotient of X is regular if X satisfies any one of the following conditions:

(i) acc X is compact;

(ii) every closed subset of X is of countable character;

(iii) $X = X_o \oplus D$, where X_o is a paracompact $< n$ - discrete space having a compatible cover $\{A_\alpha \mid \alpha < W(n)\}$ consisting of n - compact subsets and D consists of points isolated in X;

(iv) $X = X_o \oplus D$, where X_o has a countable compatible cover consisting of countably compact, sequential, normal subsets and D consists of points isolated in X.

Proof. Conditions (i) and (iii) are respectively [23, Theorem 2.1] and Theorem 10.9. Proposition 4.4 combined with Theorem 4.5 shows the sufficiency of condition (ii).

To see that condition (iv) suffices, it is enough to show that every quotient image of X_o is regular. Let $f : X_o \to Y$ be a quotient mapping and suppose $X_o = \bigcup_{n=1}^{\infty} K_n$ where each K_n is a countably compact, sequential, normal subspace and a set F is closed in X_o whenever $F \cap K_n$

is closed for each n. We first establish that X_o is sequential. To this end, let A be a subset of X_o which is not closed. Then $A \cap K_m$ is not closed for some $m \in N$. But K_m is sequential, so there exists a sequence (x_n) in $A \cap K_m$ converging to a point $x \in K_m - A$. All the more, then, (x_n) is a sequence in A converging to a point not in A, and we conclude that X_o is sequential.

Now observe that, using Proposition 4.1, the restriction of f to each K_n is a closed map. Hence by Lemma 10.8, $\{f(K_n) \mid n \in N\}$ is a countable, closed compatible cover of Y by normal subspaces. It now follows from a result of Morita (see Corollary 10.4) that Y is normal.

Part (a) of the next proposition improves [23, Theorem 2.2], which states that a necessary condition for every Hausdorff quotient of a first countable space X to be regular is that $\operatorname{acc} X$ is locally countably compact. Incidentally, the example given in [23, Example 2.3] to show that this condition is not sufficient for first countable spaces is incorrect. The first countable space X given there satisfies condition (iii) of the previous proposition (with $n = \mathcal{N}_o$), so that every quotient of X is regular. However, that $\operatorname{acc} X$ being locally countably compact is not sufficient for every quotient of a first countable space X to be regular is evident from the following example together with Proposition 13.3 (a): let X be the closed upper half plane with each point $(x,0)$ retaining its usual neighbourhood system and every other point being isolated.

The first part of the next proposition is just a restatement of Lemma 10.17. It is easily checked that the proof of Lemma 10.16 actually verifies Proposition 13.3 (b), a strengthened version of Lemma 10.16. Observe that all sequential paracompact spaces satisfy the hypothesis of the strengthened version given below.

Proposition 13.3. Each of the following statements provides a necessary condition for every quotient of a space X to be regular.

(a) If X is of point-countable type, then $\operatorname{acc} X$ is countably compact or else X is locally countably compact.

(b) If X is paracompact and each point x of a dense subset of $\operatorname{acc} X$ has the property that for every neighbourhood U of x, there is a compact set K such that $x \in K \subset U$ and such that some sequence in $U - K$ is eventually in every neighbourhood of K, then $\operatorname{acc} X$ is Lindelöf.

Corollary 13.4. If X is a paracompact space of point-countable type, then every Hausdorff quotient of X is regular iff $\operatorname{acc} X$ is compact, or else X is locally compact and $\operatorname{acc} X$ is Lindelöf.

Condition (v) in the next proposition follows from Theorem 8.7.

Proposition 13.5. Every hereditarily quotient Hausdorff image of X is regular if X satisfies any one of conditions (i) - (iv) of Proposition 13.2, or

(v) X is normal and $X = X_0 \oplus D$, where $|\beta X_0 - X_0| < \mathcal{N}_0$ and D

consists of points isolated in X.

Condition (vi) in the next proposition is Lemma 11.1, while condition (vii) follows from the fact that every bi-quotient image of a locally compact space is locally compact.

Proposition 13.6. Every bi-quotient Hausdorff image of X is regular if X satisfies any one of conditions (i) - (v) or either of the following:

(vi) X is a locally countably compact, sequential, normal space;

(vii) X is locally compact.

The following three propositions are restatements of Corollary 11.3, Lemma 11.5 and Theorem 11.8 respectively.

Proposition 13.7. Every open Hausdorff image of X is regular if X satisfies any one of conditions (i) - (vii) or either of the following:

(viii) X is of point-countable type and $\operatorname{acc} X$ is countably compact;

(ix) X is a locally countably compact, sequential space of point-countable type.

Proposition 13.8. If X is a 0-dimensional space of point-countable type and every open Hausdorff image of X is regular, then $\operatorname{acc} X$ is countably compact or else X is locally countably compact.

Corollary 13.9. If X is a 0-dimensional sequential space of point-countable type, then every open Hausdorff image of X is regular iff

acc X is countably compact or else X is locally countably compact.

Proposition 13.10 [CH]. If X is a 0-dimensional paracompact space of point-countable type, then the following statements are equivalent:

(a) every bi-quotient Hausdorff image of X is normal;

(b) every open Hausdorff image of X is normal;

(c) acc X is compact, or else X is locally compact and acc X is Lindelöf.

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FOOTNOTES

1. Credit for the descriptive terms "projector" and "resolvant" goes to Dr. Stephen Willard.

2. A space X is bi-sequential [29, Defintion 3.D.1] iff whenever a filter base F accumulates at x in X, then there is a decreasing sequence (A_n) in X which meshes with F and converges to x.

3. A space X is a k'-space [5, Defintion 3.2] iff whenever $x \in \overline{A}$ in X, then $x \in \overline{A \cap K}$ for some compact $K \subset X$.

4. A space X is a paracompact M-space iff there exists a perfect map from X onto a metrizable space.

5. Wicke actually states his result for paracompact p-spaces in the sense of Arhangel'skii [3]. But as pointed out by Michael [29, p. 92], a Hausdorff space is a paracompact M-space iff it is a paracompact p-space.

6. A property P of a space is said to be closed-hereditary iff every closed subset of a space satisfying P also satisfies P.

7. A space X is meta-Lindelöf iff every open cover of X has a point-countable open refinement.

8. I wish to thank Dr. L. Friedler for bringing the article by Alster [2] to my attention.